Elements of Probability Theory

(Baye's Theorem, Random Variables)

7.1 PROBABILITY

Probability is a concept which numerically measure the degree of uncertainity and therefore, of certainity of the occurrence of events.

If an event A can happen in m ways, and fail in n ways, all these ways being equally likely to occur, then the probability of the happening of A is

$$= \frac{\text{Number of favourable cases}}{\text{Total number of mutually exclusive and equally likely cases}} = \frac{m}{m+n}$$

and that of its failing is defined as $\dfrac{n}{m+n}$

If the probability of the happening $= p$

and the probability of not happening $= q$

then $$p+q = \frac{m}{m+n} + \frac{n}{m+n} = \frac{m+n}{m+n} = 1 \text{ or } p+q = 1$$

For instance, on tossing a coin, the probability of getting a head is $\frac{1}{2}$.

7.2 DEFINITIONS

1. **Die :** It is a small cube. Dots are . .. ... :: ::. ::: marked on its faces. Plural of the die is dice. On throwing a die, the outcome is the number of dots on its upper face.
2. **Cards :** A pack of cards consists of four suits *i.e.* Spades, Hearts, Diamonds and Clubs. Each suit consists of 13 cards, nine cards numbered 2, 3, 4, ..., 10, and Ace, a King, a Queen and a Jack or Knave. Colour of Spades and Clubs is black and that of Hearts and Diamonds is red. Kings, Queens, and Jacks are known as *face* cards.
3. **Exhaustive Events or Sample Space :** The set of all possible outcomes of a single performance of an experiment is exhaustive events or sample space. Each outcome is called a sample point. In case of tossing a coin once, $S = (H, T)$ is the *sample space.* Two outcomes Head and Tail constitute an exhaustive event because no other outcome is possible.
4. **Random Experiment :** There are experiments, in which results may be altogether different, even though they are performed under identical conditions. They are known as random experiments. Tossing a coin or throwing a die is random experiment.
5. **Trail and Event :** Performing a random experiment is called a trial and outcome is termed as event. Tossing of a coin is a trial and the turning up of head or tail is an event.
6. **Equally likely events:** Two events are said to be '*equally likely*', if one of them cannot be

expected in preference to the other. For instance, if we draw a card from well-shuffled pack, we may get any card, then the 52 different cases are equally likely.

7. **Independent event :** Two events may be *independent*, when the actual happening of one does not influence in any way the probability of the happening of the other.

   Example. The event of getting head on first coin and the event of getting tail on the second coin in a simultaneous throw of two coins are independent.

8. **Mutually Exclusive events:** Two events are known as *mutually exclusive*, when the occurrence of one of them excludes the occurrence of the other. For example, on tossing of a coin, either we get head or tail, but not both.

9. **Compound Event :** When two or more events occur in composition with each other, the simultaneous occurrence is called a compound event. When a die is thrown, getting a 5 or 6 is a compound event.

10. **Favorable Events :** The events, which ensure the required happening, are said to be favourable events. For example, in throwing a die, to have the even numbers, 2, 4 and 6 are favourable cases.

11. **Conditional Probability :** The probability of happening an event A, such that event B has already happened, is called the conditional probability of happening of A on the conditio that B has already happened. It is usually denoted by $P(A/B)$.

12. **Odds in favour of an event and odds against an event**

    If number of favourable ways = m, number of not favourable events = n

    (*i*) Odds in favour of the event = $\frac{m}{n}$, Odds against the event = $\frac{n}{m}$.

Types of Probability

1. **Classical Definition of Probability.** If there are N equally likely, mutually, exclusive and exhaustive of events of an experiment and m of these are favourable, then the probability of the happening of the event is defined as $\frac{m}{N}$.

Experimental Probability

Experimental probability is calculated by performing an activity, such as on tossing a coin the number of heads turning up and the number of trials are noted .

$$\text{Experimental probability} = \frac{\text{Number of heads turning up}}{\text{Number of trials}}$$

Example 1. *Find the probability of throwing*

(a) 5, (b) an even number with an ordinary six faced die.

Solution. (*a*) There are 6 possible ways in which the die can fall and there is only one way of throwing 5.

$$\text{Probability} = \frac{\text{Number of favourable ways}}{\text{Total number of equally likely ways}} = \frac{1}{6}$$ **Ans.**

(*b*) Total number of ways of throwing a die = 6

Number of ways falling 2, 4, 6 = 3

$$\text{The required probability} = \frac{3}{6} = \frac{1}{2}$$ **Ans.**

Example 2. *Find the probability of throwing 9 with two dice.*

Solution. Total number of possible ways of throwing two dice $= 6 \times 6 = 36$

Number of ways getting 9. *i.e.*, $(3 + 6), (4 + 5), (5 + 4), (6 + 3) = 4$.

$\therefore$ The required probability $= \dfrac{4}{36} = \dfrac{1}{9}$. **Ans.**

Example 3. *From a pack of 52 cards, one is drawn at random. Find the probability of getting a king.*

Solution. A king can be chosen in 4 ways.

But a card can be drawn in 52 ways.

$\therefore$ the required probability $= \dfrac{4}{52} = \dfrac{1}{13}$ **Ans.**

EXERCISE 7.1

1. In a class of 12 students, 5 are boys and the rest are girls. Find the probability that a student selected will be a girl. **Ans.** $\dfrac{7}{12}$

2. A bag contains 7 red and 8 black balls. Find the probability of drawing a red ball. **Ans.** $\dfrac{7}{15}$

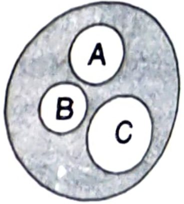
3. Three of the six vertices of a regular hexagon are chosen at random. Find the probability that the triangle with three vertices is equilateral. **Ans.** $\dfrac{1}{10}$

... will contain 53 Sundays.

7.3 ADDITION LAW OF PROBABILITY

If $p_1, p_2, \ldots p_n$ be separate probabilities of mutually exclusive events, then the probability P, that any of these events will happen is given by $P = p_1 + p_2 + p_3 + \ldots + p_n$

Proof. Let A, B, C,...... be the events, where probabilities are respectively $p_1, p_2, \ldots p_n$.



Let n be the total number of favourable cases to either A or B or C or....

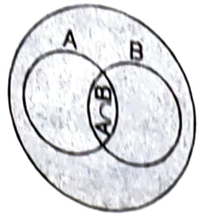
$$= m_1 + m_2 + m_3 + \ldots\ldots + m_n$$

Hence, $\quad P(A + B + C\ldots) = \dfrac{m_1 + m_2 + m_3 + \ldots + m_n}{n}$

$$= \frac{m_1}{n} + \frac{m_2}{n} + \frac{m_3}{n} + \ldots\ldots \frac{m_n}{n}$$

$$= P(A) + P(B) + P(C) + \ldots\ldots$$

$$P = p_1 + p_2 + p_3 + \ldots + p_n \qquad \textbf{Proved.}$$



Note Mutually Exclusive Events

Consider the case where two events A and B are not mutually exclusive. The probability of the event that either A or B or both occur is given as

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example 4. *An urn contains 8 black and 12 white balls. Find the probability of drawing two balls of the same colour at a time*

Solution. Probability of drawing two black balls $= \dfrac{^8C_2}{^{20}C_2}$

$\therefore$ Probability of drawing two white balls $= \dfrac{^{12}C_2}{^{20}C_2}$

$\therefore$ Probability of drawing two balls of the same colour

$$= \frac{^8C_2}{^{20}C_2} + \frac{^{12}C_2}{^{20}C_2} = \frac{\frac{8\times7}{1\times2}}{\frac{20\times19}{1\times2}} + \frac{\frac{12\times11}{1\times2}}{\frac{20\times19}{1\times2}}$$

$$= \frac{28}{190} + \frac{66}{190} = \frac{94}{190} = \frac{47}{95} \qquad \textbf{Ans.}$$

Example 5. *A bag contains four white and two black balls and a second bag contains three of each colour. A bag is selected at random, and a ball is then drawn at random from the bag chosen. What is the probability that the ball drawn is white ?.*

Solution. There are two mutually exclusive cases,
(*i*) when the first bag is chosen, (*ii*) when the second bag is chosen.

Now the chance of choosing the first bag is $\frac{1}{2}$ and if this bag is chosen, the probability of drawing a white ball is 4/6. Hence the probability of drawing a white ball from first bag is

$$\frac{1}{2} \times \frac{4}{6} = \frac{1}{3}$$

Similarly the probability of drawing a white ball from second bag is

$$\frac{1}{2} \times \frac{3}{6} = \frac{1}{4}$$

Since the events are mutually exclusive the required probability

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Example 6. *Three machines I, II and III manufacture respectively 0.4, 0.5 and 0.1 of the total production. The percentage of defective items produced by I, II and III is 2, 4 and 1 per cent respectively. For an item chosen at random, what is the probability it is defective ?*

Solution. The defective item produced by machine I

$$= \frac{0.4 \times 2}{100} = \frac{0.8}{100}$$

The defective item produced by machine II $= \dfrac{0.5 \times 4}{100} = \dfrac{2}{100}$

The defective item produced by machine III $= \dfrac{0.1 \times 1}{100} = \dfrac{0.1}{100}$

The total defective items produced by machines I, II, III

$$= \frac{0.8}{100} + \frac{2}{100} + \frac{0.1}{100} = \frac{2.9}{100} = 0.029$$

The required probability $= \dfrac{0.029}{1} = 0.029$ **Ans.**

7.4 MULTIPLICATION LAW OF PROBABILITY

If there are two independent events the respective probabilities of which are known, then the probability that both will happen is the product of the probabilities of their happening respectively.

$$P(AB) = P(A) \times P(B)$$

Proof. Suppose A and B are two independent events. Let A happen in m_1 ways and fail in n_1 ways.

$$\therefore \quad P(A) = \frac{m_1}{m_1 + n_1}$$

Also let B happen in m_2 ways and fail in n_2 ways.

$$\therefore \quad P(B) = \frac{m_2}{m_2 + n_2}$$

Now there are four possibilities

A and B both may happen, then the number of ways $= m_1 . m_2$.

A may happen and B may fail, then the number of ways $= m_1 . n_2$

A may fail and B may happen, then the number of ways $= n_1 . m_2$

A and B both may fail, then the number of ways $= n_1 . n_2$

Thus, the total number of ways $= m_1 m_2 + m_1 n_2 + n_1 m_2 + n_1 n_2$

$$= (m_1 + n_1)(m_2 + n_2)$$

Hence the probabilities of the happening of both A and B

$$P(AB) = \frac{m_1 m_2}{(m_1 + n_1)(m_2 + n_2)} = \frac{m_1}{m_1 + n_1} . \frac{m_2}{m_2 + n_2}$$

$$= P(A) . P(B)$$ **Proved.**

Example 7. *The probability that machine A will be performing an usual function in 5 years' time is $\frac{1}{4}$, while the probability that machine B will still be operating usefully at the end of the same period, is $\frac{1}{3}$*

Find the probability in the following cases that in 5 years time :

(i) Both machines will be performing an usual function.

(ii) Neither will be operating.

(iii) Only machine B will be operating.

(iv) At least one of the machines will be operating.

Solution. P (A operating usefully) $= \frac{1}{4}$, so $q\ (A) = 1-\frac{1}{4}=\frac{3}{4}$

P (B operating usefully) $= \frac{1}{3}$, so $q\ (B) = 1-\frac{1}{3}=\frac{2}{3}$

(*i*) P (Both A and B will operate usefully) $= P(A).\,P(B)$

$$= \left(\frac{1}{4}\right)\times\left(\frac{1}{3}\right)=\frac{1}{12}$$

(*ii*) P (Neither will be operating) $= q(A)\,.\,q(B) = \left(\frac{3}{4}\right)\left(\frac{2}{3}\right) = \frac{1}{2}$

(*iii*) P (Only B will be operating) $= \text{P (B)} \times q\text{ (A)} = \left(\frac{1}{3}\right)\times\left(\frac{3}{4}\right) = \frac{1}{4}$

(*iv*) P (At least one of the machines will be operating)

$= 1 - \text{P (none of them operates)}$

$= 1-\frac{1}{2}=\frac{1}{2}$ **Ans.**

Example 8. *There are two groups of subjects one of which consists of 5 science and 3 engineering subjects and the other consists of 3 science and 5 engineering subjects. An unbiased die is cast. If number 3 or number 5 turns up, a subject is selected at random from the first group, otherwise the subject is selected at random from the second group. Find the probability that an engineering subject is selected ultimately.*

(*A.M.I.E.T.E., Summer 2000*)

Solution. Probability of turning up 3 or 5 $= \frac{2}{6}=\frac{1}{3}$

Probability of selecting engineering subject from first group $= \frac{3}{8}$

Now the probability of selecting engineering subject from first group on turning up 3 or 5

$$= \left(\frac{1}{3}\right)\times\left(\frac{3}{8}\right)=\frac{1}{8} \qquad \text{... (1)}$$

Probability of not turning up 3 or 5 $= 1-\frac{1}{3}=\frac{2}{3}$

Probability of selecting engineering subject from second group $= \frac{5}{8}$

Now probability of selecting engineering subject from second group on not turning up 3 or 5

$$=\frac{2}{3}\times\frac{5}{8}=\frac{5}{12} \qquad \cdots (2)$$

Probability of the selection of engineering subject $=\frac{1}{8}+\frac{5}{12}$ [From (1) and (2)]

$$=\frac{13}{24}$$

Ans.

Example 9. *An urn contains nine balls, two of which are red, three blue and four black. Three balls are drawn from the urn at random. What is the probability that*

(i) the three balls are of different colours?

(ii) the three balls are of the same colour? (A.M.I.E., Summer 2000)

Solution.

Urn contains 2 Red balls, 3 Blue balls and 4 Black balls.

(*i*) Three balls will be of different colours if one ball is red, one blue and one black ball are drawn.

$$\text{Required probability} = \frac{{}^2C_1 \times {}^3C_1 \times {}^4C_1}{{}^9C_3} = \frac{2\times3\times4}{84}=\frac{2}{7}$$

Ans.

(*ii*) Three balls will be of the same colour if either 3 blue balls or 3 black balls are drawn.

P (3 Blue balls or 3 Black balls) $= P$ (3 Blue balls) $+ P$ (3 Black balls)

$$=\frac{{}^3C_3}{{}^9C_3}+\frac{{}^4C_3}{{}^9C_3}=\frac{1+4}{84}=\frac{5}{84}$$

Ans.

Example 10. *An urn A contains 2 white and 4 black balls. Another urn B contains 5 white and 7 black balls. A ball is transferred from the urn A to the urn B, then a ball is drawn from urn B. Find the probability that it is white.*

Solution. Urn A contains 2 white and 4 black balls.

Urn B contains 5 white and 7 black balls.

Now there are two cases of transferring a ball from A to B.

Case I. When a white ball is transferred from A to B

$$P\,(\text{Transfer of a white ball}) = \frac{2}{2+4}=\frac{1}{3}$$

After transfer of a white ball, urn B contains 6 white balls and 7 black balls.

P (Drawing a white ball from urn B after transfer)

$= P$ (Transfer of a white ball) $\times$ P (Drawing of a white ball)

$$=\left(\frac{1}{3}\right)\left(\frac{6}{6+7}\right)=\frac{1}{3}\times\frac{6}{13}=\frac{2}{13}$$

Case II. When a black ball is transferred from A to B.

$$P\,(\text{Transfer of a black ball}) = \frac{4}{2+4}=\frac{2}{3}$$

After transfer of a black ball, urn B contains 5 white and 8 black balls.

P (Drawing a white ball from urn B after transfer)

$=$ P (Transfer of a black ball) $\times$ P (Drawing of a white ball)

$$=\frac{2}{3}\left(\frac{5}{5+8}\right)=\frac{10}{39}$$

Required probability $= \frac{2}{13} + \frac{10}{39} = \frac{16}{39}$ **Ans.**

Example 11. *A bag contains 10 white and 15 black balls. Two balls are drawn in succession. What is the probability that first is white and second is black ?*

Solution. Probability of drawing one white ball $= \frac{10}{25}$

Probability of drawing one black ball without replacement = 15/24

Required probability of drawing first white ball and second black ball

$$= \frac{10}{25} \times \frac{15}{24} = \frac{1}{4}$$ **Ans.**

Example 12. *A committee is to be formed by choosing two boys and four girls out of a group of five boys and six girls. What is the probability that a particular boy named A and a particular girl named B are selected in the committee?*

Solution. Two boys are to be selected out of 5 boys. A particular boy A is to be included in the committee. It means that only 1 boy is to be selected out of 4 boys.

Number of ways of selection $= {}^4C_1$

Similarly a girl B is to be included in the committee.

Then only 3 girls are to be selected out of 5 girls.

Number of ways of selection $= {}^5C_3$

$$\text{Required probability} = \frac{{}^4C_1 \times {}^5C_3}{{}^5C_2 \times {}^6C_4} = \frac{4 \times 10}{10 \times 15} = \frac{4}{15}$$ **Ans.**

Example 13. *Three groups of children contain respectively 3 girls and 1 boy; 2 girls and 2 boys; 1 girl and 3 boys. One child is selected at random from each group. Find the chance of selecting 1 girl and 2 boys.*

Solution. There are three ways of selecting 1 girl and two boys.

I way : Girl is selected from first group, boy from second group and second boy from third group.

Probability of the selection of (Girl + Boy + Boy)

$$= \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{18}{64}$$

II way : Boy is selected from first group, girl from second group and second boy from third group.

Probability of the selection of (Boy + Girl + Boy)

$$= \frac{1}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{6}{64}$$

III way : Boy is selected from first group, second boy from second group and the girl from the third group.

Probability of selection of (Boy + Boy + Girl)

$$= \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4} = \frac{2}{64}$$

$$\text{Total probability} = \frac{18}{64} + \frac{6}{64} + \frac{2}{64} = \frac{26}{64} = \frac{13}{32}$$ **Ans.**

Example 14. *The number of children in a family in a region are either 0, 1 or 2 with probability 0.2, 0.3 and 0.5 respectively. The probability of each child being a boy or girl 0.5. Find the probability that a family has no boy.*

Solution. Here there are three types of families

(*i*) Probability of zero child (boys) = 0.2

(*ii*)

Boy	Girl
0	1
1	0

Probability of zero boy in case II

$$= 0.3 \times 0.5 = 0.15$$

(*iii*)

Boy	Girl
0	2
1	1
2	0

In this case probability of zero boy $= 0.5 \times \frac{1}{3} = 0.167$

Considering all the three cases, the probability of zero boy

$$= 0.2 + 0.15 + 0.167 = 0.517$$

Ans.

Example 15. *A husband and wife appear in an interview for two vacancies in the same post. The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that*

(*i*) *both of them will be selected.*

(*ii*) *only one of them will be selected, and*

(*iii*) *none of them will be selected ?*

Solution. P (husband's selection) $= \frac{1}{7}$, P (wife's selection) $= \frac{1}{5}$

(*i*) P (both selected) $= \frac{1}{7} \times \frac{1}{5} = \frac{1}{35}$

(*ii*) P (only one selected) = P (only husband's selection) + P (only wife's selection)

$$= \frac{1}{7} \times \frac{4}{5} + \frac{1}{5} \times \frac{6}{7} = \frac{10}{35} = \frac{2}{7}$$

(*iii*) P (none of them will be selected) $= \frac{6}{7} \times \frac{4}{5} = \frac{24}{35}$

Ans.

Example 16. *A problem of statistics is given to three students A, B and C whose chances of solving it are $\frac{1}{2}$, $\frac{3}{4}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved?*

Solution. The probability that A can solve the problem $= \frac{1}{2}$

The probability that A cannot solve the problem $= 1-\frac{1}{2}$.

Similarly the probability that B and C cannot solve the problem are

$$\left(1-\frac{3}{4}\right) \text{ and } \left(1-\frac{1}{4}\right)$$

$\therefore$ The probability that A, B, C cannot solve the problem

$$=\left(1-\frac{1}{2}\right)\times\left(1-\frac{3}{4}\right)\times\left(1-\frac{1}{4}\right)=\frac{1}{2}\times\frac{1}{4}\times\frac{3}{4}=\frac{3}{32}$$

Hence, the probability that the problem can be solved

$$=1-\frac{3}{32}=\frac{29}{32}$$ **Ans.**

Example 17. *A student takes his examination in four subjects α, β, γ, δ. He estimates his chances of passing in α as $\frac{4}{5}$, in β as $\frac{3}{4}$, in γ as $\frac{5}{6}$ and in δ as $\frac{2}{3}$. To qualify, he must pass in α and at least two other subjects. What is the probability that he qualifies?* (AMIETE, Dec. 2010)

Solution. $P(\alpha)=\frac{4}{5}$, $P(\beta)=\frac{3}{4}$, $P(\gamma)=\frac{5}{6}$, $P(\delta)=\frac{2}{3}$

There are four possibilities of passing at least two subjects.

(*i*) Probability of passing β, γ and failing δ

$$=\frac{3}{4}\times\frac{5}{6}\times\left(1-\frac{2}{3}\right)=\frac{3}{4}\times\frac{5}{6}\times\frac{1}{3}=\frac{5}{24}$$

(*ii*) Probability of passing γ, δ and failing β

$$=\frac{5}{6}\times\frac{2}{3}\times\left(1-\frac{3}{4}\right)=\frac{5}{6}\times\frac{2}{3}\times\frac{1}{4}=\frac{5}{36}$$

(*iii*) Probability of passing δ, β and failing γ

$$=\frac{2}{3}\times\frac{3}{4}\times\left(1-\frac{5}{6}\right)=\frac{2}{3}\times\frac{3}{4}\times\frac{1}{6}=\frac{1}{12}$$

(*iv*) Probability of passing β, γ, δ

$$=\frac{3}{4}\times\frac{5}{6}\times\frac{2}{3}=\frac{5}{12}$$

Probability of passing at least two subjects.

$$=\frac{5}{24}+\frac{5}{36}+\frac{1}{12}+\frac{5}{12}=\frac{61}{72}$$

Probability of passing α and at least two subjects.

$$=\frac{4}{5}\times\frac{61}{72}=\frac{61}{90}$$ **Ans.**

Example 18. *There are 6 positive and 8 negative numbers. Four numbers are chosen at random, without replacement, and multiplied. What is the probability that the product is a positive number ?*

7.5 CONDITIONAL PROBABILITY

Let A and B be two events of a sample space S and let $P(B) \neq 0$. Then conditional probability of the event A, given B, has already occured denoted by $P(A/B)$, is defined by

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \qquad \text{... (1)}$$

Theorem. If the events A and B defined on a sample space S of a random experiment are independent, then

$$P(A/B) = P(A) \text{ and } P(B/A) = P(B)$$

Proof. A and B are given to be independent events,

$$P(A \text{ and } B) = P(A) \bullet P(B)$$

$$\Rightarrow \qquad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A).P(B)}{P(B)} = P(A)$$

$$\Rightarrow \qquad P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B).P(A)}{P(A)} = P(B)$$

7.6 BAYE'S THEOREM

If $B_1, B_2, B_3, \ldots\ldots B_n$ are mutually exclusive events with $P(B_i) \neq 0$, $(i = 1, 2, \ldots n)$ of a random experiment then for any arbitrary event A of the sample space of the above experiment with $P(A) > 0$, we have

$$P(B_i / A) = \frac{P(B_i)P(A/B_i)}{\sum_{i=1}^{n} P(B_i)P(A/B_i)}$$ (for $n = 3$)

$$P(B_2 / A) = \frac{P(B_2)P(A/B_2)}{P(B_1)P(A/B_1)+P(B_2)P(A/B_2)+P(B_3)P(A/B_3)}$$

Proof. Let S be the sample space of the random experiment.

The events $B_1, B_2, \ldots, B_n$ being exhaustive

$$S = B_1 \cup B_2 \cup \ldots \cup B_n$$

$$[\because A \subset S]$$

$$A = A \cap S$$
$$= A \cap (B_1 \cup B_2 \cup \ldots \cup B_n)$$
$$= (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n) \quad \text{[Distributive Law]}$$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_n)$$
$$= P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + \ldots + P(B_n)P(A/B_n)$$
$$= \sum_{i=1}^{n} P(B_i)P(A/B_i) \quad \ldots (1)$$

Now, $\quad P(A \cap B_i) = P(A)P(B_i / A)$

$$\Rightarrow \quad P(B_i / A) = \frac{P(A \cap B_i)}{P(A)} = \frac{P(B_i)P(A/B_i)}{\sum_{i=1}^{n} P(B_i)P(A/B_i)} \quad \text{[Using (1)]}$$

Note. $P(B)$ is the probability of occurrence B. If we are told that the event A has already occurred. On knowing about the event A, $P(B)$ is changed to $P(B/A)$. With the help of Baye's theorem we can calculate $P(B / A)$.

Example 23. *An urn I contains 3 white and 4 red balls and an urn II contains 5 white and 6 red balls. One ball is drawn at random from one of the urns and is found to be white. Find the probability that it was drawn from urn I.*

Solution. Let U_1: the ball is drawn from urn I

U_2 : the ball is drawn from urn II

W : the ball is white.

We have to find $P(U_1/W)$

By Baye's Theorem

$$P(U_1 / W) = \frac{P(U_1)P(W/U_1)}{P(U_1)P(W/U_1) + P(U_2)P(W/U_2)} \quad \ldots (1)$$

Since two urns are equally likely to be selected, $P(U_1) = P(U_2) = \frac{1}{2}$

$P(W/U_1) = P$ (a white ball is drawn from urn I) $= \frac{3}{7}$

$P(W/U_2) = P$ (a white ball is drawn from urn II) $= \frac{5}{11}$

$\therefore$ From (1), $P(U_1/W) = \dfrac{\frac{1}{2}\times\frac{3}{7}}{\frac{1}{2}\times\frac{3}{7}+\frac{1}{2}\times\frac{5}{11}} = \dfrac{33}{68}$ **Ans.**

Example 24. *Three urns contains 6 red, 4 black; 4 red, 6 black; 5 red, 5 black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red find the probability that it is drawn from the first urn.*

Solution. Let U_1: the ball is drawn from U_1.

U_2: the ball is drawn from U_2.

U_3: the ball is drawn from U_3.

R: the ball is red.

We have to find $P(U_1/R)$.

By Baye's Theorem,

$$P(U_1/R) = \frac{P(U_1)P(R/U_1)}{P(U_1)P(R/U_1)+P(U_2)P(R/U_2)+P(U_3)P(R/U_3)} \quad \ldots(1)$$

Since the three urns are equally likely to be selected $P(U_1) = P(U_2) = P(U_3) = \frac{1}{3}$

Also $P(R/U_1) = P$ (a red ball is drawn from urn I) $= \frac{6}{10}$

$P(R/U_2) = P$ (a red ball is drawn from urn II) $= \frac{4}{10}$

$P(R/U_3) = P$ (a red ball is drawn from urn III) $= \frac{5}{10}$

$\therefore$ From (1), we have $P(U_1/R) = \dfrac{\frac{1}{3}\times\frac{6}{10}}{\frac{1}{3}\times\frac{6}{10}+\frac{1}{3}\times\frac{4}{10}+\frac{1}{3}\times\frac{5}{10}} = \dfrac{2}{5}$ **Ans.**

Example 25. *In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total. If their output 5, 4 and 2 per cent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine B?*

Solution. A: bolt is manufactured by machine A.

B: bolt is manufactured by machine B.

C: bolt is manufactured by machine C.

$P(A) = 0.25, P(B) = 0.35, P(C) = 0.40$

The probability of drawing a defective bolt manufactures by machine A is $P(D/A) = 0.05$
Similarly, $P(D/B) = 0.04$ and $P(D/C) = 0.02$
By Baye's theorem

$$P(B/D) = \frac{P(B)P(D/B)}{P(A)P(D/A)+P(B)P(D/B)+P(C)P(D/C)}$$

$$= \frac{0.35\times0.04}{0.25\times0.05+0.35\times0.04+0.40\times0.02} = 0.41$$ **Ans.**

EXERCISE 7.3

1. There are 4 boys and 2 girls in room A and 5 boys and 3 girls in room B. A girl from one room of the two rooms laughed loudly. What is the probability that the girl who laughed was from room B.

Ans. $\frac{9}{17}$

2. A coin is tossed. It turns up heads and two balls will be drawn from urn A othersise two balls will be drawn from urn B. Urn A contains 3 black and 5 white balls. Urn B contains 7 black and 1 white ball. In both cases selection are to be made with replacement. What is the probability that urn A is used given that both the balls drawn are black.

Ans. 0.125

3. Assume that a factory has two machines. Past records show that machine A produces 30% of the items of output and machine B produces 70% of the items. Further 5% of the items produced by the machine A were defective and only 10% produced by the machine B were defective. If a defective item is drawn at random. what is the probability that it was produced by machine B.

Ans. $\frac{7}{22}$

4. A box A contains 1 red and 2 white marbles and another box B contains 3 red and 2 white marbles. One marble is drawn at random from one of the boxes and it was found to be red. Find the probability that it was drawn from the box B.

Ans. $\frac{9}{14}$

5. A factory produces a certain type of output by three types of machines. The respective daily production figures are machine I : 3,000 units; machine II: 2,500 units and machine III: 4,500 units.

   Past experience shows that 1% of the output produced by machine I is defective. The corresponding percentage of defectives for the other two machines are 1.2% and 2% respectively. An item is drawn at random from the daily production and is found to be defective. What is the probability that it comes from the output of (a) Machine I (b) Machine II and (c) Machine III?

Ans. (*a*) 0.2, (*b*) 0.2, (*c*) 0.6

6. TV components are produced by two machines A and B. 50% of the components are produced by machine A, with an estimated of 10% of them being defective. machine B 20% of the components produced are defective. If a component taken at random is found to be defective. What is the probability that the component was produced by machine A.

Ans. $\frac{1}{3}$